# STEADY UNIFORM FLOWS

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Uniform flows are open-channel flows, which do not change with distance at a particular time. The flow condition can be obtained via momentum concept. In this chapter, qualifications and the momentum concept for the uniform flow are given. Then, three widely-used formulas are introduced. The origin of these formulas are traced mechanistically and related coefficients are discussed.

## 1. Qualifications for Uniform Flow

The uniform flow has the following features:

(1) the depth, water area, velocity, and discharge at every section of the channel reach are constant.

(2) the slope of energy line ($S_e$), the slope of water surface ($S_w$), and the slope of the channel bottom ($S_o$) are the same.

A constant velocity may be interpreted as a constant time-averaged velocity, i.e., This should mean that the flow possesses a constant velocity at every point on the channel section within the channel reach. That is, the velocity distribution across the channel section is unaltered in the reach. Such a stable pattern is attained when the boundary layer is fully developed. Uniform flow is considered to be steady only because unsteady uniform flow is practically nonexistent.

**2. Shear Stress in Uniform Flow**

A uniform flow is developed if the gravity force is balanced by the resistance. Consider the force balance in the uniform flow (momentum approach). The gravity force acting on the fluid element is given by

$$F_g = \gamma A dx \sin\theta \tag{1}$$

The resisting force due to shear along the wetted perimeter is

$$F_f = \tau_0 P dx \tag{2}$$

Equating Eq.(1) and Eq.(2) results in

$$\tau_0 = \gamma R_h S_0 \tag{3}$$

where $S_0$ denotes the channel slope ($= \tan\theta \approx \sin\theta$). Note that $S_0 = S_w = S_e$ for uniform flows and Eq.(3) is valid for any channels of arbitrary cross sections. From Eq.(3), the friction slope is defined by

$$S_f = \frac{\tau_0}{\gamma R_h}$$

which is not the same as $S_0$ ($S_w$ or $S_e$) for flows other than the uniform flow. The friction slope means that the slope is obtained from the momentum concept.

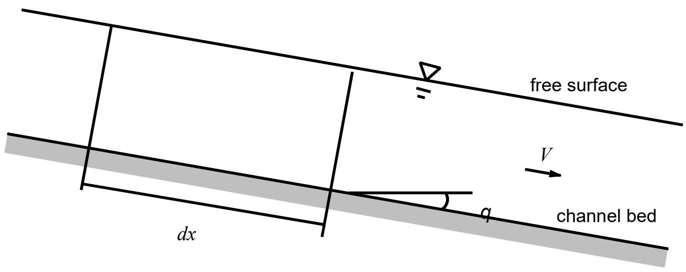



Figure 1. Force balance of the flow in a prismatic open channel

## 3. Uniform Flow Formulas

### 3.1 Chezy Formula (1769)

The history of uniform flow begins with Antoine de Chezy with data on a canal upstream of Paris. Chezy's formula takes the form of

$$V = C\sqrt{R_h S_0} \tag{4}$$

where $V$ = mean velocity, $R_h$ = hydraulic radius, $S_0$ = bed slope, and $C$ = Chezy coefficient representing the wall roughness. It is seen that $C$ is of [$L^{1/2}/T$].

Chezy's formula can be derived theoretically. In fluid mechanics, the shear stress at the bottom is represented by

$$\tau_0 = c_f \rho \frac{V^2}{2}$$

where $c_f$ is a geometric factor influenced by the boundary roughness (or flow resistance coefficient). That is, the resisting force is obtained by assuming that the force per unit stream bed area is proportional to the squared mean velocity. Then, the total resisting force is

$$F_f = KV^2 P dx \tag{5}$$

where $K$ is a proportionality. From Eqs.(1) and (5), it is obtained that $V = C\sqrt{R_h S_0}$ with $C = \sqrt{\gamma / K}$. Therefore, it can clearly be understood that Chezy's formula was proposed based on the momentum concept.

### 3.2 Manning Formula (1889)

For incompressible, steady flows at a constant depth (uniform flow) in a prismatic open-channel, the Manning formula is widely used. Substituting $C = C_m R_h^{1/6} / n$ into Eq.(5) leads to Manning's formula such as

$$V = \frac{C_m}{n} R_h^{2/3} S^{1/2} \tag{6}$$

where the value of $C_m$ is 1 and 1.49 for SI and USC units, respectively. This happens because Manning's equation is dimensionally non-homogeneous. The following Table delivers representative Manning's roughness coefficients for various boundary materials.

Actually, Philippe Gauckler proposed the same type of formula, very similar to Eq.(6) three years earlier (Hager, 2015). In order to take Gauckler's contribution into account, Eq.(6) is also called Gauckler-Manning formula. Later, Strickler proposed the following formula:

$$V = 21.1\left(2gS_0R_h\right)^{1/2}\left(\frac{R_h}{d_m}\right)^{1/6} \tag{7}$$

where $d_m$ is the median sediment size representing the size of equivalent spheres, of which the surface of the channel is composed. It is interesting to note that Eq.(7), Manning-Strickler's formula, is dimensionally homogeneous.

Table 1. Manning's Roughness Coefficient (Chow, 1959)

| Material | Typical Manning roughness coefficient |
|---|---|
| Concrete | 0.012 |
| Gravel bottom with sides — concrete | 0.020 |
| — mortared stone | 0.023 |
| — riprap | 0.033 |
| Natural stream channels | |
| Clean, straight stream | 0.030 |
| Clean, winding stream | 0.040 |
| Winding with weeds and pools | 0.050 |
| With heavy brush and timber | 0.100 |
| Flood Plains | |
| Pasture | 0.035 |
| Field crops | 0.040 |
| Light brush and weeds | 0.050 |
| Dense brush | 0.070 |
| Dense trees | 0.100 |

### 3.3 Darcy-Weisbach Formula

If Chezy coefficient ($C$) is replaced by $\sqrt{8g/f}$, then the following Darcy-Weisbach formula is obtained:

$$V = \sqrt{\frac{8g}{f}}\sqrt{R_h S} \tag{7}$$

where the roughness coefficient *f* is given by

$$f = fn(k / R_h, \text{Re}) \tag{8}$$

in which *k* is the roughness height. Values of the roughness coefficient *f* are given in the Moody diagram which is obtained from experiments of pipe flows. The expressions for *f* are

$$f = \frac{24}{\text{Re}} \qquad \text{Re} \le 500 \tag{9a}$$

$$f = \frac{0.223}{\text{Re}^{1/4}} \qquad 500 < \text{Re} \le 25{,}000 \tag{9b}$$

For fully-developed turbulent flows over hydraulically-smooth boundary with $\text{Re} > 25{,}000$

$$\frac{1}{\sqrt{f}} = 2\log \text{Re}\sqrt{f} + 0.4 \qquad \text{Re} > 25{,}000 \tag{9c}$$

and for fully-developed turbulent flows over hydraulically-rough boundary with $u_* k / \nu > 70$ or $\text{Re}\sqrt{f} / (R / k) > 50$,

$$\frac{1}{\sqrt{f}} = 2\log \frac{R_h}{k} + 2.16 \qquad \text{Re} > 25{,}000 \tag{9d}$$

where *k* is the equivalent size of the Nikuradse type surface roughness and $u_*$ is the shear velocity $(= \sqrt{\tau_0 / \rho})$. Note that the roughness is no longer a function of Re for fully- developed turbulent flows over hydraulically rough surface.

Among three resistance factors, the Darcy-Weisbach $f$ has the best theoretical background. It is non-dimensional, and its values for steady uniform flows are given in the Moody diagram.

However, it should be emphasized that the roughness coefficient $f$ in Darcy-Weisbach formula is a local quantity as indicated by the above relationship whereas the roughness coefficients in the other formulas are reach-averaged quantities. The reason why Darcy-Weisbach formula is not so popular in the practical hydraulics is that the roughness coefficient comes from the pipe flow experiments. That is, there is no Moody diagram for the open-channel flow, and $f-\mathrm{Re}$ relationship changes according to channel geometry.

### 3.4 Dimensional Consideration

Manning's, Chezy's, and Darcy-Weisbach's formulas were originally developed empirically although theoretical numerous attempts were made later. From the three relationships, it is clear that

$$\sqrt{\frac{8}{f}}=\frac{C}{\sqrt{g}}=\frac{C_m}{\sqrt{g}}\frac{R_h^{1/6}}{n} \tag{10}$$

The above equation reveals that

(a) The Chezy $C$ has the dimension of $\sqrt{g}$.

(b) $C_m$ in the Manning's formula has the dimension of $\sqrt{g}$ because it is unreasonable to assume $n$ changes with changing $g$. Therefore, the Manning's $n$ has the dimension of $[\mathrm{L}^{1/6}]$.

Although $n$ has a dimension of $[\mathrm{L}^{1/6}]$, in practice the same numerical value of $n$ is used in English system as in SI system, and hence the constant 1.49 absorbs not only the dimension of $g$ but also the conversion factor from SI system.

### 3.5 Variation of *n* with Surface Roughness

It is a well known fact that for a given rough surface the value of *n* hardly changes with the depth or discharge of the flow provided the roughness elements of the channel are statistically homogeneous and randomly distributed. It is clear that

$$\frac{n}{k^{1/6}} = f(R_h / k, \mathrm{Re}) \tag{11}$$

As can be seen in Eq.(9d), only for fully-developed turbulent flows over a rough boundary, $n / k^{1/6}$ is a function of $R_h / k$ but not *Re*. If *n* is truly a measure of the surface roughness, $n / k^{1/6}$ will be a constant independent of either $R_h / k$ or *Re*. Strickler (1923) proposed that

$$\frac{n}{k^{1/6}} = 0.0342 \tag{12}$$

To investigate the variation of *n* with the flow depth, Eqs.(9d) and (10) can be combined to give

$$\frac{R_h^{1/6}}{n} = \frac{\sqrt{8g}}{C_m}\left(2\log\frac{12R_h}{k}\right) \tag{13}$$

or

$$\frac{n}{k^{1/6}} = \frac{C_m}{\sqrt{8g}}\frac{(R_h / k)^{1/6}}{2\log(12R_h / k)} \tag{14}$$

Rouse (1946) plotted Eq.(13) to demonstrate the insensitivity of Manning's *n* to $R_h$. This plot is shown in Fig. 2 ($R_h / k$ versus $R_h^{1/6} / n$) together with Stricker's formula and a straight line with a 6:1 slope passing through the point corresponding to the minimum value of $n / k^{1/6}$. The figure indicates that Eq.(13) coincides well with Strickler's formula, meaning that $n$ is not sensitive to $R_h$.

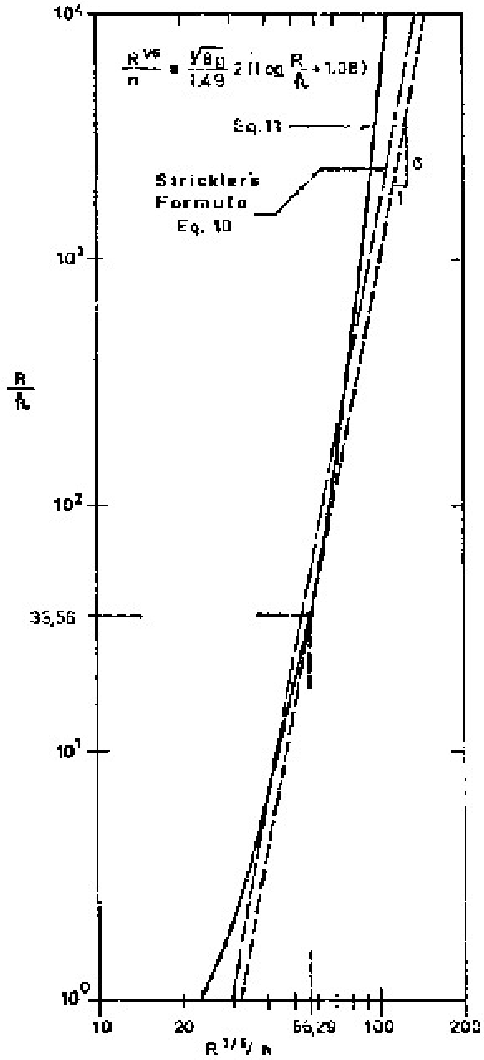



Figure 2. Variation of $R_h^{1/6}/n$ with $R_h/k$

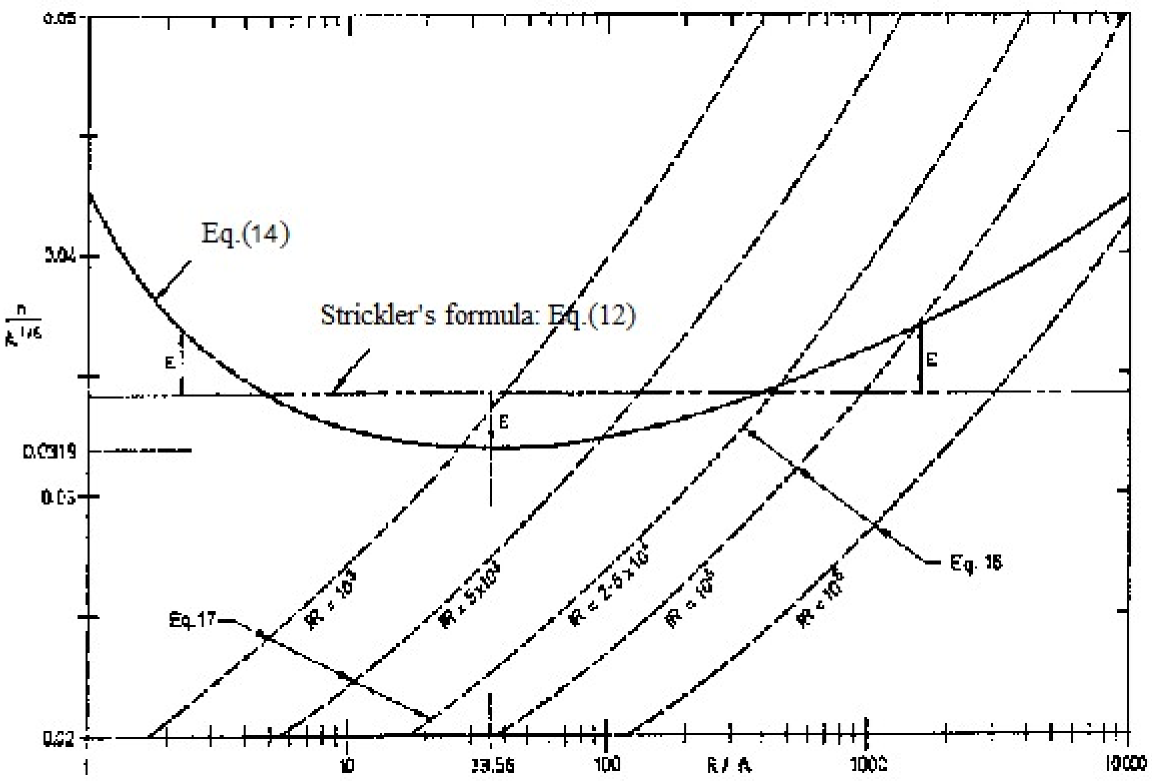



Figure 3. Variation of $n / k^{1/6}$ with $R_h / k$

An alternative plot, perhaps more straightforward, to show the small variation of n with respect to $R_h$ is given in Fig. 3 as done by Chow (1959, p.206). Figure 3 shows the change of $n / k^{1/6}$ with $R_h / k$. In the figure, Eq.(14) is plotted with Strickler's formula. It can be seen that *n* changes little with $R_h$. Note also that a thousandfold change in *k* results in a threefold change in *n*. This means that *k* is much more sensitive than *n*. Therefore, it can be said if sands in the channel bed are uniform, Manning's *n* is constant along the channel, *f* changes a little and *C* changes significantly. Plotted also are similar relations for fully developed flow for hydraulically smooth boundary (denoted by Eq.16) and for flow in the transition region (denoted by Eq.17). In such case, n is a function of $R_h / k$ and Reynolds number.

**4. Conveyance**

When either the Chezy formula or Manning formula is used for uniform flow computation, the discharge becomes

$$Q = K\sqrt{S} \tag{15}$$

and the conveyance is

$$K = Q / \sqrt{S} \tag{16}$$

The conveyance is a measure of carrying capacity of the channel section. When Chezy formula is used, the conveyance is

$$K = CA\sqrt{R_h} \tag{17}$$

and when Manning formula is used,

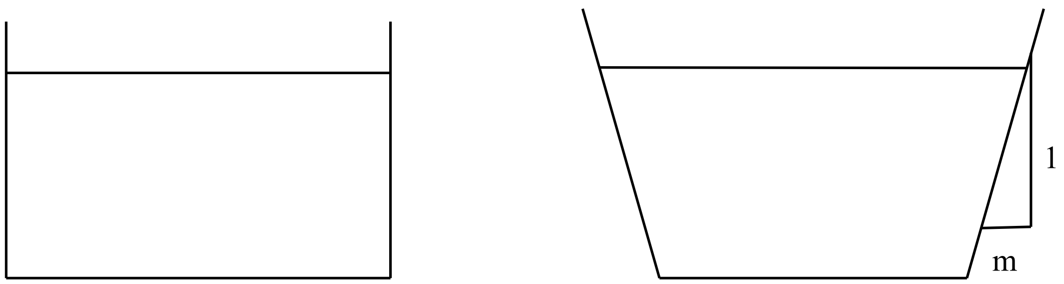
$$K = \frac{C_m}{n} A R_h^{2/3} \tag{18}$$

**5. Best Hydraulic Sections**

What is the best hydraulic cross section in designing the channel (not analyzing the channel)? Some channel cross sections are more efficient than others in that they provide more area for a given wetted parameter. Or the sections have the least wetted perimeter for a given cross-sectional area.

(1) Rectangular Cross Section

With the help of Manning's equation, if $Q$, $n$, and $S$ are known, the cross sectional area can be expressed as a function of perimeter $P$ as




Channel cross sections

$$A = cP^{2/5} \tag{19}$$

where $c$ is known. For a rectangular channel, the width ($B$) is $P-2h$, so

$$(P-2h)h = cP^{2/5} \tag{20}$$

Differentiating eq.(19) with respect to $y$ results

$$\left(\frac{dP}{dh}-2\right)h+(P-2h)=\frac{2}{5}cP^{-3/5}\frac{dP}{dh} \tag{21}$$

Setting $dP/dh = 0$ gives $P = 4h$, or

$$b = 2h \tag{22}$$

That is, the best rectangular hydraulic section has the depth which is one-half of the width. (Q) Obtain the best hydraulic section of a rectangular cross section with a free board $F$ at both sides.

(2) Trapezoidal Cross Section

For the trapezoidal section, similarly,

$$bh + mh^2 = (P - 2h\sqrt{1+m^2})h + mh^2 = cP^{2/5} \tag{23}$$

Differentiating eq.(22) with respect to $h$ yields

$$P = 4h\sqrt{1+m^2} - 2mh \tag{24}$$

Now, for a constant water depth ($h$), $m$ is to be sought. That is, $dP/dm = 0$ leads to

$$m = 1/\sqrt{3} \tag{25}$$

Therefore, the best hydraulic section is one-half of a hexagon.

(Q) Best hydraulic section in terms of sediment transport

**6. Flows in Composite Roughness Channel and Compound Channel**

Consider the channel section of composite roughness. Various methods for computing the equivalent roughness are available. A simple method proposed by Horton (1993) assumes that the velocities in the sub-sections are the same as the average velocity over the whole cross section. That is, the mean velocity in the $i$-th section is given by

$$V_i = \frac{1}{n_i}\left(\frac{A_i}{P_i}\right)^{2/3} S^{1/2}$$

The total area is expressed as

$$A = \sum_i A_i = \sum_i P_i n_i^{3/2} \frac{V^{3/2}}{S^{3/4}}$$

From Manning's formula, the total area is also given by

$$A = \frac{V^{2/3} P n^{2/3}}{S^{3/4}}$$

Equating the relationships for the areas leads to the equivalent roughness coefficient such as

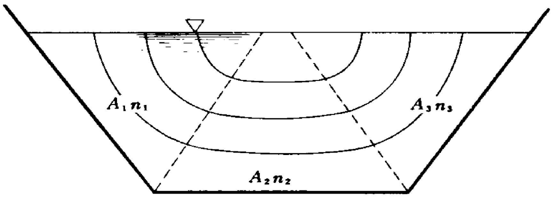
$$n = \frac{\left( \sum_i P_i n_i^{3/2} \right)^{2/3}}{P^{2/3}} \tag{26}$$

Similarly, for compound channel section, the mean velocity is given by

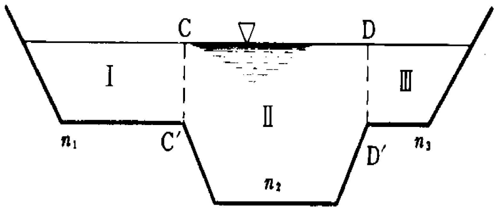
$$V_i = \frac{1}{n_i} \left( \frac{A_i}{P_i} \right)^{2/3} S^{1/2}$$

Thus the total discharge is given by

$$Q = \sum_i A_i V_i \tag{27}$$




(a) channel with different roughnesses




(b) compound channel section

It should be noted that composite roughness can be used in assessing the discharge when the velocity structure is relatively homogeneous over the entire section. Otherwise, mean velocity should be estimated at each section such as in a compound channel.

**References**


Chow, V.T. (1959). *Open-Channel Hydraulics.* McGraw Hill Book Company, New York, NY.

Horton, R.A. (1933). Separate roughness coefficient for channel bottom and sides, Engineering News Records, 111(2).

Sabersky, R.H., Acosta, A.J., and Hauptmann, E.G. (1971). *Fluid Flow*. Macmillan, New York, NY.

**Problems**

1. Show that if sands in the channel bed is uniform, Manning's $n$ is constant along the channel, Darcy Weisbach's $f$ changes a little and Chezy's $C$ changes significantly.

2. Obtain the best hydraulic section of a rectangular cross section with a free board $F$ at both sides.

3. In general, the wide open-channel can safely defined as a rectangular channel whose width is greater than 10-15 times the depth of flow. That is,

$$B / y = 10 - 15$$

where $B$ is the width and $y$ is the flow depth. In the wide open channel, the dynamics due to the circulations in the direction transverse to the main flow direction can be ignored. Consider the (rectangular-shaped) open channel at the hydraulic laboratory in Yonsei University. The width of the channel is about 1 m, and the flow depth of $y$ = 0.25 m is going to be maintained. The side wall is made of glass ($n$ = 0.01), and the channel bottom is covered by the concrete block ($n$ = 0.03 assumed) to supply extra roughness. Can this channel be considered as a wide rectangular open channel?

4. Derive the governing equation for long wave theory which can be applied to many problems in open-channel flows by averaging the following continuity and momentum equations:

$$\nabla \cdot \vec{V} = 0$$

$$\frac{d\vec{V}}{dt} = -\frac{1}{\rho}\nabla p^*$$

where $p^* = p + \gamma z$. Explain why the wave celerity in the long wave theory is $\sqrt{gh}$.